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Editorial

K-theory is a new discipline of mathematics embracing concepts and problems central to many other major disciplines of mathematics. The aim of this journal is to provide a forum for the presentation, discussion, and critical evaluation of significant advances in the mathematical sciences which are related to *K*-theory. It is expected that this will bring together work having close conceptual and methodological relationships which, hitherto, had been scattered in the literature.

The success of *K*-theory rests with its many applications to important problems in other disciplines and its ability to adapt to ongoing research in various areas of mathematics after obtaining a foothold there.

Some spectacular successes of *K*-theory include its contributions to solving the following problems: Congruence subgroup problem for classical groups, metaplectic problem for classical groups, structure of finite central simple algebras, Serre's problem for polynomial rings, cancellation and other stability theorems, induction theorems over finite groups, structure and computation of Witt groups, Riemann–Roch theorems, structure and computation of Chow groups, generalizations of class field theory, generalizations of Bott periodicity, vector fields on spheres problem, Hopf invariant one problem, Adams' conjecture, index theorems, structure and computation of obstruction groups in topology, classification of exotic spheres, triangulation of manifolds, rigidity of $K(\pi_1)$-manifolds, topological invariance of Pontryagin classes, de Rham's conjecture, Novikov's conjecture, spherical space form problem, realization of scalar curvature, extension theory for C^*-algebras, glueing theory for dynamical systems, Kadison's conjecture, and existence of discrete series representations.

The close bonds *K*-theory has with diverse parts of mathematics have continually brought it fresh impulses for new growth and development. In this process, *K*-theory has assimilated large tracts of other disciplines including, for example, number theory, ring and module theory, quadratic form theory, algebraic geometry, algebraic and differential topology, and operator algebra theory. On the other hand, simultaneous applications of *K*-theory in various disciplines have promoted the transfer of concepts, methodologies, and information amongst them, thus tightening the bonds between them. Through assimilation and crosspollination, *K*-theory has become a unifying force in mathematical research.

The interplay *K*-theory affords between various mathematical disciplines can be illustrated by looking at the history of the subject. Glancing at the list of applications above, one sees that *K*-theory treats concepts and problems, many of which were firmly established over a half-century ago. Missing, however, at that time was a satisfactory awareness of structural similarities amongst diverse disciplines. The 1940's and 50's witnessed an increasing perception of such similarities, particularly because of the influence of algebraic topology, and the creation of a formal language for expressing them, namely category theory. Here, notions of structure are stated in terms of categories having certain properties and categories are compared via the concept of a functor. Categories occur naturally by grouping together mathematical objects 'of the same kind', such as vector bundles over a given space or quadratic forms over a given ring. Some important ways of studying objects 'of the same kind' are to attach structure-preserving invariants to them, such as Chern classes in the case of vector bundles, or to combine them in a pairwise fashion to form new objects of the same kind, such as taking the orthogonal sum of two quadratic forms. K-theory was created by using category theory to provide a systematic basis for the considerations above.

In his paper *Classes de faisceaux et théorème de Riemann–Roch*, completed in several letters to Serre in 1957, Grothendieck associates to each (small) category $\mathscr{C}$, a group $K(\mathscr{C})$ defined as the free Abelian group on the isomorphism classes of objects of $\mathscr{C}$ modulo relations given by the structure being preserved. This was the first paper in K-theory. Concerning his group and the choice of the letter K, Grothendieck says in his letter of 9 February 1985 to Bruce Magurn:

> The way I first visualized a K-group was as a group of 'classes of objects' of an Abelian (or more generally, additive) category, such as coherent sheaves on an algebraic variety, or vector bundles, etc. I would presumably have called this group $C(X)$ (X being a variety or any other kind of 'space'), C the initial letter of 'class', but my past in functional analysis may have prevented this, as $C(X)$ designates also the space of continuous functions on X (when X is a topological space). Thus, I reverted to K instead of C, since my mother tongue is German, Class = Klasse (in German), and the sounds corresponding to C and K are the same.

Grothendieck's results for K-groups and his formulation and proof of the Riemann–Roch theorem for algebraic vector bundles were the first results and applications of K-theory, although the rubric K-theory would not be coined until several years later.

Grothendieck's K-construction was applied around 1960 to topological vector bundles and it was shown that the sequence of functors $K^{-n} = K\Sigma^n$, where $\Sigma^n(X)$ denotes the n-fold suspension of a topological space X, leads to a generalized cohomology theory. This theory was christened topological K-theory. Some early triumphs of topological K-theory were the solution to the vector fields on spheres problem (1962) and the first index theorem (1963).

Drawing on a natural, structural equivalence between the category of topological vector bundles on a compact Hausdorff space X and the category of finitely generated projective modules over the ring $C(X)$, mathematicians attempted constructing algebraic analogues of the K-groups $K^{-n}(X)$ of topological K-theory, for any ring A. At first, only two functors K_0 and K_1 were constructed. These functors, together with certain exact sequences and stability theorems, generalizing ones in topological K-theory, were announced in 1962. The new subject was dubbed algebraic K-theory. Its first major problems were Serre's problem for polynomial rings, posed already in 1955, and the problem of finding the higher K_n's, i.e., constructing $K_n(A)$ for all $n \geqslant 0$. Some early, outstanding achievements of algebraic K-theory were the solution to the congruence subgroup problem for certain families of classical groups and computations of Siebenmann–Wall and Whitehead obstruction groups in topology. The machinery of algebraic K-theory was used, in the case of obstruction groups arising from spaces with finite fundamental group, to bring the methodology of classical induction theory for finite group and tools of number theory to bear on the computations.

In 1966, the correct definition of algebraic K_2 was found and concepts of algebraic K-theory were applied to quadratic and Hermitian forms, eventually giving rise to the K-theory of forms. Research in this area was stimulated right from the beginning by its close ties to geometric surgery on manifolds and the problem of computing surgery obstruction groups, where great strides were made in the following decade. Negative algebraic K-theory groups $K_{-n}(A)$ for all $n \geqslant 0$ were uncovered in 1967 and found a decade later important applications in geometric topology such as in the triangulation of stratified spaces, which led in turn to new formulations of negative K-groups, and in the solution of de Rham's conjecture for linear versus topological equivalence of rotations.

The major problems of algebraic K-theory of the 1960's were solved in the 1970's. About 1970, several approaches to constructing higher algebraic K-theory groups were proposed, but it was first in 1972 after proper categorical-geometrical foundations were laid and $K_n(\mathscr{C})$ was defined for any category $\mathscr{C}$ having suitable structure and any $n \geqslant 0$, that a satisfactory theory was established. The higher K-theory of categories extended that of the Grothendieck group $K(\mathscr{C})$ $(=K_0(\mathscr{C}))$ in a natural way and allowed introducing higher K-theory into all areas where the Grothendieck group had played a role including algebraic K-theory (which is the K-theory of rings), the K-theory of forms, and algebraic geometry via the K-theory of schemes. The higher K-theory of schemes has become a valuable tool in the study of Chow groups. Towards the end of the 1970's, étale methods in algebraic geometry influenced the development of the étale K-theory of rings and schemes. The other major problem of algebraic K-theory, namely Serre's problem, was solved in 1976 and later generalized to regular rings and forms.

Connections found in the 1960's between 'classical' algebraic K-theory and other areas of mathematics were extended in the 1970's to higher K-theory. One example concerns the functions of arithmetic. In the late 1960's, Birch and Tate conjectured a connection between the order of K_2 of an arithmetic Dedekind ring and the value of the Dedekind zeta function at -1. In the early 1970's, Lichtenbaum extended the conjecture to one between higher K-groups of an arithmetic Dedekind ring and values of the zeta function at negative integers. The Birch–Tate conjecture was proved at all odd primes and special cases of the Lichtenbaum conjecture have been established. As time progressed, the higher K-theory of rings and schemes was linked in other ways to zeta functions and to L-functions, regulator maps, and other functions of arithmetic and currently, there are many conjectures in this area.

Higher K-theory was used in the latter half of the 1970's in developing 'higher' class field theory for multi-dimensional local fields, for surfaces over finite fields, and for arithmetic surfaces. Ties between the K-theory of fields and Galois cohomology, uncovered at that time, led in the current decade to spectacular results on the structure of finite central simple algebras.

Towards the end of the 1970's, another major expansion of K-theory took shape and was again influenced by structural similarities between different categories. Using a natural duality between the homotopy category of compact Hausdorff spaces and the homotopy category of commutative C^*-algebras, mathematicians developed the K-theory of (not necessarily commutative) C^*-algebras, generalizing topological K-theory, and the KK-theory of C^*-algebras, a bivariant theory containing K-theory in one variable and its dual K-homology in the other. Important applications of the new theories have been made in and outside of C^*-algebra theory. Inside, for example, K-theoretic invariants are frequently the only tools available for distinguishing one C^*-algebra from another and K-theory was used in solving Kadison's conjecture that $C^*_{\mathrm{red}}\ (F_n)$ has no nontrivial projections. KK-theory used to classify extensions of C^*-algebras. Outside of C^*-algebra theory, KK-theory was used in establishing cases of the Novikov conjecture on the homotopy invariance of higher signatures and in describing how the irreducible components of a topological dynamical system are glued together to form a global system. In recent years, the algebraic aspects of KK-theory have been studied and have led on the one hand to cyclic cohomology, which turns out to be closely related to algebraic K-theory, and on the other hand to research aimed at constructing KK-groups for rings other than C^*-algebras. Such research is included in the current issue.

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